

Minimal surfaces in contact Sub - Riemannian manifolds.

Nataliya Shcherbakova

Abstract

In the present paper we consider generic Sub-Riemannian structures on the co-rank 1 non-holonomic vector distributions and introduce the associated canonical volume and "horizontal" area forms. As in the classical case, the Sub-Riemannian minimal surfaces can be defined as the critical points of the "horizontal" area functional. We derive an intrinsic equation for minimal surfaces associated to a generic Sub-Riemannian structure of co-rank 1 in terms of the canonical volume form and the "horizontal" normal. The presented construction permits to describe the Sub-Riemannian minimal surfaces in a generic Sub-Riemannian manifold and can be easily generalized to the case of non-holonomic vector distributions of greater co-rank.

The case of contact vector distributions, in particular the $(2, 3)$ -case, is studied more in detail. In the latter case the geometry of the Sub-Riemannian minimal surfaces is determined by the structure of their characteristic points (i.e., the points where the hyper-surface touches the horizontal distribution) and characteristic curves. It turns out that the known (see [4]) classification of the characteristic points of the Sub-Riemannian minimal surfaces in the Heisenberg group H^1 holds true for the minimal surfaces associated to a generic contact $(2, 3)$ distribution. Moreover, we show that in the $(2, 3)$ case the Sub-Riemannian minimal surfaces are the integral surfaces of a certain system of ODE in the extended state space. In some particular cases the Cauchy problem for this system can be solved explicitly. We illustrate our results considering Sub-Riemannian minimal surfaces in the Heisenberg group and the group of roto-translations.

1 Introduction

In the classical Riemannian geometry minimal surfaces realize the critical points of area functional with respect to variations that preserve the boundary of a given domain. The Sub-Riemannian minimal surfaces are the natural

generalization of the classical ones in Sub-Riemannian manifolds known also as the Carnot-Carathéodory spaces. The notion of a minimal surface in the Sub-Riemannian manifold was introduced in [6] in the framework of Geometric Measure Theory, and then was studied in [7], [8], [9], [10], [5], [4], [11]. The main part of the results of the cited papers are related to the Heisenberg group H^1 , though recently the first steps were done toward the analysis of the group of roto-translations ([5], [10]). A very fruitful geometrical model was recently proposed in [4]. The authors gave a general geometrical definition of the Sub-Riemannian minimal surfaces by means of CR-structures in 3-dimensional pseudohermitian manifolds and studied in great detail the case of the Heisenberg group H^1 . In our paper we propose an alternative (with respect to [4]) coordinate-free way to define Sub-Riemannian minimal surfaces using the tools of Sub-Riemannian geometry¹.

Our general construction is the following. Denote by M an n -dimensional smooth manifold and let Δ be a co-rank 1 smooth vector distribution on it (“horizontal” distribution). Assume that Δ is endowed with a Riemannian structure, which can be described by fixing an orthonormal basis of vector fields X_1, \dots, X_{n-1} on Δ (see [3]). Then we say that Δ defines a Sub-Riemannian structure on M , and M is a Sub-Riemannian manifold. It turns out that there is a canonical way to define a volume form $\mu \in \Lambda^n M$ associated to the Sub-Riemannian structure of M . Moreover, in analogy with the classical Riemannian case one can define the Sub-Riemannian normal of a hyper-surface $W \subset M$ as a unit vector field ν such that

$$\int_{\Omega} i_{\nu} \mu = \max_{\substack{X \in \Delta \\ \|X\|_{\Delta}=1}} \int_{\Omega} i_X \mu, \quad \Omega \subset W.$$

The $n-1$ -form $i_{\nu} \mu$ is the Sub-Riemannian analog of the classical area form on M . It is easy to see that this definition correlates perfectly with the classical definition of the Riemannian normal and area since any Riemannian manifold is a Sub-Riemannian manifold with $\Delta \equiv TM$.

As in the classical case, one can define the Sub-Riemannian minimal surfaces in M as the critical points of the functional associated to the Sub-Riemannian area form. It turns out that these surfaces satisfy the following intrinsic equation

$$(d \circ i_{\nu} \mu) \Big|_W = 0.$$

The described construction opens a wide possibility to study the Sub-Riemannian minimal surfaces associated to generic Sub-Riemannian struc-

¹For the detailed exposition on Sub-Riemannian geometry the reader can consult [1], [2], [3].

tures of any dimension. Moreover, it does not require the existence of any additional global structure in M . It worth to mention that in the case of the Heisenberg group, as well as in the case of the roto-translational group, our definition coincides with the known ones (see [7], [8], [9], [5] and references therein)

In the second part of this paper we consider more in detail the case of contact Sub-Riemannian structures, in particular the case of 2-dimensional distributions in 3-dimensional manifolds. In the latter case all information related to the intrinsic geometry of Sub-Riemannian minimal surfaces is encoded in the *characteristic curve* $\gamma : [0, T] \mapsto W$ such that $\dot{\gamma}(t) \in T_{\gamma(t)}W \cap \Delta_{\gamma(t)}$ for all $t \in [0, T]$. The tangent of this curve is orthogonal to the Sub-Riemannian normal of W and it is defined (as well as the Sub-Riemannian area and normal ν) away from the *characteristic points of W* , where $T_q W \equiv \Delta_q$. Applying the classical method of characteristics to the minimal surface equation we show that the Sub-Riemannian minimal surfaces are actually the integral surfaces of a certain system of ODE in the extended space $M \times S^1$. The characteristic points are either the singular points of these surfaces or the singularities of their projection on the base manifold. It turns out that the classification of the characteristic points described in [4] (Theorem B) for the Heisenberg case, holds true for any $(2, 3)$ contact Sub-Riemannian structure.

We conclude our analysis comparing the characteristic curves of a Sub-Riemannian minimal surface with the Sub-Riemannian geodesics in M . We show that in general the characteristic curves do not coincide with the Sub-Riemannian geodesics, though in some particular cases they do, as for example in the case of the Heisenberg group, while in the group of roto-translations only a certain class of characteristic curves are Sub-Riemannian geodesics.

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2 Sub-Riemannian minimal surfaces: general construction

2.1 Sub-Riemannian structures and associated objects

Let M be an n -dimensional smooth manifold. Consider a co-rank 1 vector distribution Δ on M :

$$\Delta = \bigcup_{q \in M} \Delta_q, \quad \Delta_q \subset T_q M, \quad q \in M.$$

By definition, the Sub-Riemannian structure on M is a pair $(\Delta, \langle \cdot, \cdot \rangle_\Delta)$, where $\langle \cdot, \cdot \rangle_\Delta$ denotes a smooth family of Euclidean inner products on Δ . In what follows we will call Δ *the horizontal distribution* and keep the same notation Δ both for the vector distribution and for the associated Sub-Riemannian structure.

Let $X_i, i = 1, \dots, n-1$ be a horizontal orthonormal basis on Δ :

$$\Delta_q = \text{span}\{X_1(q), \dots, X_{n-1}(q)\}, \quad q \in M,$$

$$\langle X_i(q), X_j(q) \rangle_\Delta = \delta_{ij}, \quad q \in M, \quad i, j = 1, \dots, n-1.$$

By $\Theta \in \Lambda^{n-1} \Delta$ we will denote the corresponding Euclidean volume form on Δ .

In what follows we will assume that Δ is *bracket-generating* on M . In the present case this means that

$$\text{span}\{X_i(q), [X_i, X_j](q), \quad i, j = 1, \dots, n-1, \quad q \in M\} = T_q M.$$

Hereafter the square brackets denote the Lie brackets of vector fields. If Δ is bracket-generating, then by the Frobenius theorem it is completely non-holonomic, i.e., there are no invariant sub-manifolds of M such that their tangent spaces coincides with Δ at any point.

There is an alternative way to define the distribution Δ as the kernel of some differential 1-form. Let $\omega \in \Lambda^1 M$ be such a form :

$$\Delta_q = \text{Ker } \omega_q = \{v \in T_q M : \omega_q(v) = 0\}, \quad q \in M.$$

In general, the form ω is defined up to a multiplication by a non-zero scalar function. It is easy to check that Δ is bracket-generating at $q \in M$ if and only if $d_q \omega \neq 0$.

By the standard construction the Riemannian structure on Δ can be extended to the spaces of forms $\Lambda^k \Delta$, $k \leq n-1$. In particular, for any 2-form σ we set

$$\|\sigma_q\|_\Delta = \left(\sum_{\substack{i,j=1 \\ i < j}}^{n-1} \sigma_q(X_i(q), X_j(q))^2 \right)^{\frac{1}{2}},$$

$\{X_i(q)\}_{i=1}^{n-1}$, as before, being an orthonormal horizontal basis of Δ_q . Now we can fix the choice of the form ω defined above by setting

$$(1) \quad \omega_q(\Delta_q) = 0, \quad \|d_q \omega\|_\Delta = 1, \quad \forall q \in M.$$

We will call the 1-form satisfying (1) *the canonical 1-form associated to Δ* . In the fixed horizontal orthonormal basis $\{X_i(q)\}_{i=1}^{n-1} \in \Delta_q$ equations (1) become

$$(2) \quad \omega_q(X_i(q)) = 0, \quad \sum_{\substack{i,j=1 \\ i < j}}^{n-1} d_q \omega(X_i(q), X_j(q))^2 = 1, \quad i = 1, \dots, n-1.$$

It is worth to note that the canonical 1-form, ω satisfying (2) is defined up to a sign and does not depend on the choice of the horizontal basis. In local coordinates in M the components of ω can be expressed in terms of the coordinates of the vector fields X_i and their first derivatives, since due to the Cartan formula

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]),$$

and hence

$$\|d_q \omega\|_\Delta^2 = \sum_{\substack{i,j=1 \\ i < j}}^{n-1} d_q \omega(X_i(q), X_j(q))^2 = \sum_{\substack{i,j=1 \\ i < j}}^{n-1} \omega_q([X_i, X_j](q))^2.$$

Once the orientation in M is fixed by choosing the sign of ω , the following volume form

$$\mu = \Theta \wedge \omega$$

is uniquely defined. We will call this volume form *the canonical volume form associated to Δ* . The canonical volume form μ is a “global” object in M , though it is intrinsically defined by the Sub-Riemannian structure on Δ .

2.2 Horizontal area form

Let $W \subset M$, $\dim W = n - 1$ be a smooth hyper-surface in M and let $\Omega \subset W$ be an open domain. For simplicity we assume that the vector field $X \in TM$ is transversal to W , though this assumption is not restrictive. Consider the flow generated by X in M :

$$\begin{aligned}\Pi^X : \quad [0, \varepsilon] \times \Omega &\mapsto M, \\ \Pi^X(t, q) &= e^{tX}(q), \quad q \in M.\end{aligned}$$

Denote by

$$(3) \quad \Pi_{(\varepsilon, \Omega)}^X = \{e^{tX}(q), \quad q \in \Omega, \quad t \in [0, \varepsilon]\}$$

the cylinder formed by the images of Ω translated along the integral curves of X parametrized by $t \in [0, \varepsilon]$. Clearly, $\Pi_{(0, \Omega)}^X \equiv \Omega$. By definition,

$$Vol(\Pi_{(\varepsilon, \Omega)}^X) = \int_{\Pi_{(\varepsilon, \Omega)}^X} \mu = \int_{[0, \varepsilon] \times \Omega} (\Pi^X)^* \mu,$$

where $(\Pi^X)^*$ denotes the pull-back map associated to Π^X and μ is the canonical volume form defined above².

Definition 1 *The following quantity*

$$(4) \quad A_\Delta(\Omega) = \max_{\substack{X \in \Delta \\ \|X\|_\Delta = 1}} \lim_{\varepsilon \rightarrow 0} \frac{Vol(\Pi_{(\varepsilon, \Omega)}^X)}{\varepsilon}$$

is called the Sub-Riemannian (or horizontal) area of the domain Ω associated to Δ .

Remark. The horizontal area defined by (4) is nothing but the generalization of the classical notion of the Euclidean area: it defines the area of the base of a cylinder as the ratio of its volume and height.

Let us find a more convenient expression for (4). First of all we observe that since $(\Pi^X)^* \mu$ is a form of maximal rank n on M we have $dt \wedge (\Pi^X)^* \mu = 0$. Hence

$$0 = i_{\partial_t} (dt \wedge (\Pi^X)^* \mu) = i_{\partial_t} dt \wedge (\Pi^X)^* \mu - dt \wedge i_{\partial_t} (\Pi^X)^* \mu,$$

²Here we use the canonical volume form associated to Δ , though the whole construction works for any volume form in M .

i.e.,

$$(\Pi^X)^*\mu = dt \wedge i_{\partial_t}(\Pi^X)^*\mu.$$

Taking into account that $\Pi_*^X \partial_t = X$ we obtain

$$\begin{aligned} A_\Delta(\Omega) &= \max_{\substack{X \in \Delta \\ \|\bar{X}\|_\Delta = 1}} \lim_{\varepsilon \rightarrow 0} \frac{\text{Vol}(\Pi_{(\varepsilon, \Omega)}^X)}{\varepsilon} = \\ &= \max_{\substack{X \in \Delta \\ \|\bar{X}\|_\Delta = 1}} \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \int_{\Pi_{(\varepsilon, \Omega)}^X} \mu = \max_{\substack{X \in \Delta \\ \|\bar{X}\|_\Delta = 1}} \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \int_{[0, \varepsilon] \times \Omega} dt \wedge i_{\partial_t}(\Pi^X)^*\mu = \\ &= \max_{\substack{X \in \Delta \\ \|\bar{X}\|_\Delta = 1}} \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \int_0^\varepsilon \left(\int_{\Pi_{(t, \Omega)}^X} i_X \mu \right) dt = \max_{\substack{X \in \Delta \\ \|\bar{X}\|_\Delta = 1}} \int_\Omega i_X \mu. \end{aligned}$$

Definition 2 *The horizontal unite vector field $\nu \in \Delta$, $\|\nu\|_\Delta = 1$ such that*

$$\int_\Omega i_\nu \mu = \max_{\substack{X \in \Delta \\ \|\bar{X}\|_\Delta = 1}} \int_\Omega i_X \mu$$

is called the Sub-Riemannian or horizontal normal of $\Omega \subset W$. The $(n-1)$ -form $i_\nu \mu$ is called the Sub-Riemannian or horizontal area form associated to Δ .

In general, the given definition of the Sub-Riemannian normal ν does not require the existence of any global structure in M (for instance, one does not need a Riemannian structure in M). Nevertheless, if M is a Riemannian manifold whose Riemannian structure is compatible with the Sub-Riemannian structure on Δ , i.e., if the inner product $\langle \cdot, \cdot \rangle$ on TM satisfies $\langle \cdot, \cdot \rangle_\Delta = \langle \cdot, \cdot \rangle|_\Delta$, then it is easy to see that the Sub-Riemannian normal ν is nothing but the projection on Δ of the Riemannian unit normal N of W , normalized w.r.t. $\|\cdot\|_\Delta$. This is the consequence of the following relation

$$\int_\Omega i_X \mu = \int_\Omega \langle X, N \rangle i_N \mu, \quad \forall X \in \text{Vec}(M).$$

Thus if $X_1, \dots, X_{n-1} \in \Delta$ is an orthonormal horizontal basis of Δ , then

$$(5) \quad \nu = \sum_{i=1}^{n-1} \nu_i X_i, \quad \nu_i = \frac{\langle N, X_i \rangle}{\sqrt{\langle N, X_1 \rangle^2 + \dots + \langle N, X_{n-1} \rangle^2}},$$

and the horizontal area form reads

$$i_\nu \mu = \langle \nu, N \rangle i_N \mu = \sqrt{\langle N, X_1 \rangle^2 + \cdots + \langle N, X_{n-1} \rangle^2} i_N \mu.$$

Now let the hyper-surface W be defined as a level set of a smooth, let us say C^2 , function:

$$W = \{q \in M : F(q) = \text{const}, F \in C^2(M), d_q F \neq 0\}.$$

Let $X \equiv X_n$ be a vector field transversal to W and such that $\{X_1(q), \dots, X_n(q)\}$ form an orthonormal basis of $T_q M$ at $q \in M$. Then

$$N(q) = D_0^{-1} \sum_{i=1}^n X_i F(q) X_i(q), \quad D_0 = \left(\sum_{i=1}^n X_i F(q)^2 \right)^{1/2}$$

and

$$(6) \quad \nu(q) = D_1^{-1} \sum_{i=1}^{n-1} X_i F(q) X_i(q), \quad D_1 = \left(\sum_{i=1}^{n-1} X_i F(q)^2 \right)^{1/2}.$$

Here $X_i F$ denotes the directional derivative of F along the vector field X_i .

2.3 Sub-Riemannian minimal surfaces

Now let us compute the variation of the horizontal area $A_\Delta(\cdot)$. Assume that $\Omega \subset W$ is a bounded domain and let $V \in \text{Vec}(M)$ be such that $V|_{\partial\Omega} = 0$. Consider a one-parametric family of hyper-surfaces generated by the vector field V

$$\Omega^t = e^{tV} \Omega, \quad \Omega^0 \equiv \Omega,$$

and denote by ν^t the horizontal unit normals to Ω^t . We have

$$A_\Delta(\Omega^t) = \int_{e^{tV} \Omega} i_{\nu^t} \mu = \int_{\Omega} (e^{tV})^* i_{\nu^t} \mu = \int_{\Omega} e^{tL_V} i_{\nu^t} \mu.$$

Further,

$$(7) \quad \frac{\partial}{\partial t} \Big|_{t=0} A_\Delta(\Omega^t) = \int_{\Omega} L_V i_\nu \mu + \int_{\Omega} i_{\frac{\partial \nu^t}{\partial t} \Big|_{t=0}} \mu.$$

It is not hard to show that the second integral in (7) vanishes, because the horizontal vector field $\frac{\partial \nu^t}{\partial t}|_{t=0}$ is tangent to Ω . Indeed, at a generic (non-characteristic) point $q \in \Omega$ we have $\nu(q) \notin T_q \Omega$ and $\dim \Delta_q \cap T_q \Omega = n - 2$. On the other hand, differentiating the equality $\langle \nu^t, \nu^t \rangle_\Delta = 1$ we get

$$(8) \quad \left\langle \frac{\partial \nu^t}{\partial t} \Big|_{t=0}, \nu \right\rangle_\Delta = 0,$$

and hence $\frac{\partial \nu^t}{\partial t}|_{t=0}(q) \in T_q \Omega$.

Further, using Cartan's formula we transform the first part of (7):

$$\int_{\Omega} L_V i_\nu \mu = \int_{\Omega} (i_V \circ d + d \circ i_V) i_\nu \mu = \int_{\Omega} (i_V \circ d \circ i_\nu) \mu + \int_{\Omega} (d \circ i_V \circ i_\nu) \mu.$$

Applying the Stokes theorem to the second integral we see that it vanishes:

$$\int_{\Omega} (d \circ i_V \circ i_\nu) \mu = \int_{\partial \Omega} (i_V \circ i_\nu) \mu = 0$$

provided $V|_{\partial \Omega} = 0$ and $\partial \Omega$ is sufficiently regular. Thus,

$$\frac{\partial}{\partial t} \Big|_{t=0} A_\Delta(\Omega^t) = \int_{\Omega} i_V \circ (d \circ i_\nu \mu).$$

Definition 3 *We say that the hyper-surface W is a minimal surface w.r.t. the Sub-Riemannian structure Δ (or just Δ -minimal) iff*

$$(9) \quad (d \circ i_\nu \mu) \Big|_W = 0.$$

We remark that the minimality of a hyper-surface does not depend on the chosen orientation in M . It is also easy to see that the whole construction can be easily generalized for the case of vector distributions of co-rank greater than 1.

2.4 Canonical form of the minimal surface equation in contact Sub-Riemannian manifolds

Let $n = 2m + 1$ and assume that Δ is a contact distribution, i.e., the $2m + 1$ -form $(d\omega)^m \wedge \omega$ is non-degenerate. Then we say that M is a contact Sub-Riemannian manifold. In the contact case there exists a unique vector field $X \in TM$ such that

$$(10) \quad \omega_q(X(q)) = 1, \quad d_q \omega(V, X(q)) = 0, \quad \forall V \in \Delta_q.$$

Such a vector field is called *the Reeb vector field* associates to the contact form ω . Using this vector field we can extend the Sub-Riemannian structure on Δ to the whole TM . The resulting Riemannian structure in M is by definition compatible with Δ . The basis of vector fields $\{X_1, \dots, X_{2m}, X\}$ is then a canonical basis associated to the contact Sub-Riemannian structure Δ .

Set $X_{2m+1} \equiv X$. Denote by $c_{ij}^k \in C^\infty(M)$ the structural constants of the frame $\{X_i\}_{i=1}^{2m+1}$

$$(11) \quad [X_i, X_j] = - \sum_{k=1}^{2m+1} c_{ij}^k X_k.$$

Let $\{\theta_i\}_{i=1}^{2m+1}$ be the basis of 1-forms dual to $\{X_i\}_{i=1}^{2m+1}$. Clearly, $\theta_{2m+1} \equiv \omega$ and the canonical volume form is

$$\mu = \theta_1 \wedge \dots \wedge \theta_{2m+1}.$$

From the Cartan formula it follows that

$$(12) \quad d\theta_k = \sum_{\substack{i,j=1 \\ i < j}}^{2m+1} c_{ij}^k \theta_i \wedge \theta_j, \quad k = 1, \dots, 2m+1.$$

Let us now derive the canonical form of the minimal surface equation (9) in contact Sub-Riemannian manifolds. First we calculate the interior product of $\nu = \sum_{i=1}^{2m} \nu_i X_i$ with the canonical volume form:

$$i_\nu \mu = \left(\sum_{k=1}^{2m} (-1)^{k+1} \nu_k \theta_1 \wedge \dots \wedge \widehat{\theta_k} \wedge \dots \wedge \theta_{2m} \right) \wedge \theta_{2m+1} = \Xi \wedge \theta_{2m+1}.$$

Here $\widehat{\theta_k}$ denotes the omitted element in the wedge product and

$$\Xi = \sum_{k=1}^{2m} (-1)^{k+1} \nu_k \theta_1 \wedge \dots \wedge \widehat{\theta_k} \wedge \dots \wedge \theta_{2m}.$$

Further,

$$d i_\nu \mu = d\Xi \wedge \theta_{2m+1} - \Xi \wedge d\theta_{2m+1}.$$

Recalling now that $d\nu_k = \sum_{i=1}^{2m+1} X_i(\nu_k)\theta_i$, we obtain

$$\begin{aligned} d\Xi \wedge \theta_{2m+1} &= \sum_{k=1}^{2m} (-1)^{k+1} \left(d\nu_k \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta_k} \wedge \cdots \wedge \theta_{2m} + \right. \\ &\quad \left. + \nu_k d(\theta_1 \wedge \cdots \wedge \widehat{\theta_k} \wedge \cdots \wedge \theta_{2m}) \right) \wedge \theta_{2m+1} = \left(\sum_{k=1}^{2m} X_k(\nu_k) + \sum_{j=1}^{2m} \nu_k c_{kj}^j \right) \mu. \end{aligned}$$

On the other hand,

$$\Xi \wedge d\theta_{2m+1} = \Xi \wedge \sum_{\substack{i,j=1 \\ i < j}}^{2m+1} c_{ij}^{2m+1} \theta_i \wedge \theta_j = - \left(\sum_{k=1}^{2m} \nu_k c_{k2m+1}^{2m+1} \right) \mu.$$

Summing up we obtain the following equation:

$$(13) \quad \left[\operatorname{div}^\Delta \nu + \sum_{i=1}^{2m} \nu_i(q) \left(\sum_{j=1}^{2m+1} c_{ij}^j \right) \right] \Big|_W = 0.$$

The left-hand side of (13) is called *the Sub-Riemannian mean curvature* of the hyper-surface W , while its first term

$$\operatorname{div}^\Delta \nu = \sum_{i=1}^{2m} X_i(\nu_i)$$

is called *the horizontal divergence* of the Sub-Riemannian normal ν . Equation (13) is the canonical equation of Sub-Riemannian minimal surfaces in a contact Sub-Riemannian manifold. In the rest of the present paper we will try to analyze it in the less-dimensional case of $m = 1$.

Remark In general the Sub-Riemannian structures are not equivalent to the CR-structures, which were used in [4] and the successive publications by other authors, and consequently in general equation (13) is different from its analog obtained in [4] for 2-dimensional minimal surfaces in 3-dimensional contact CR manifolds. Nevertheless, in some particular cases, like the Heisenberg group and the group of roto-translations, both models produce the same result.

Example 1 (*The Heisenberg distribution*) Let $M = \mathbb{R}^{2m+1}$ and denote by $(x_1, \dots, x_{2m}, t) = q$ the Cartesian coordinates in M . Let Δ be such that $\Delta_q = \text{span}\{X_i(q)\}_{i=1}^{2m}$, $q \in M$, where

$$(14) \quad X_i(q) = \partial_{x_i} + \frac{x_{i+m}}{2} \partial_t,$$

$$X_{i+m}(q) = \partial_{x_{i+m}} - \frac{x_i}{2} \partial_t, \quad i = 1, \dots, m.$$

The vector distribution Δ is characterized by the following commutative relations:

$$(15) \quad [X_i, X_j] = 0, \quad \text{for } j \neq i + m, \quad [X_i, X_{i+m}] = -\partial_t,$$

and therefore it is a co-rank 1 bracket-generating distribution. The vector fields X_i , $i = 1, \dots, 2m$, generates the so-called *Heisenberg Lie algebra* on \mathbb{R}^{2m+1} . In what follows we will call the vector distributions which satisfy the commutative relations (15) *the Heisenberg distribution* and denote it by Δ^{H^m} . The space \mathbb{R}^{2m+1} endowed with the structure of this distribution is called the Heisenberg group H^m .

By solving (2) we find the canonical 1-form ω :

$$(16) \quad \omega = \pm \frac{1}{\sqrt{m}} \left(dt - \frac{1}{2} \sum_{i=1}^m (x_{i+m} dx_i - x_i dx_{i+m}) \right),$$

and correspondingly the Reeb vector field $X = \pm \sqrt{m} \partial_t$. Clearly ω is a contact form since $(d\omega)^m \wedge \omega = \pm \frac{1}{m^m} \bigwedge_{i=1}^{2m} dx_i \wedge dt$ in non-degenerate. The only non-zero structural constants of the canonical frame are $c_{ii+m}^{2m+1} = \pm \frac{1}{\sqrt{m}}$, $i = 1, \dots, m$. Due to the high degeneracy of the Sub-Riemannian structure the canonical minimal surface equation takes a very simple form:

$$\text{div}^{\Delta^{H^m}} \nu \Big|_W = 0.$$

This is the well known minimal surface equation in the Heisenberg group (see [7], [8], [4], [11], etc.)

3 Sub-Riemannian minimal surfaces for (2, 3) contact vector distributions

In this section we analyze the case of a contact distribution Δ of rank 2 in the 3-dimensional manifold M . In this case the intrinsic information

about the geometry of the Δ -minimal surface W is encoded in the so-called *characteristic curves* of W , which can be defined as the leaves of the one-dimensional foliation $TW \cap \Delta$. The singular points of the characteristic curves are called *the characteristic points*. At these points Δ is tangent to W and hence the Sub-Riemannian normal (as well the horizontal area form) is not defined.

In the case of $(2, 3)$ contact distributions the characteristic curve, being a one-dimensional sub-manifold, has no intrinsic invariants. However, one can extract some information about the global geometry of the Δ -minimal surfaces by analyzing the type of its characteristic points.

Let $n = 3$ and assume that Δ is such that $\Delta_q = \text{span}\{X_1(q), X_2(q)\}$, $q \in M$. Set $X_3 \equiv X$, where X is the Reeb vector field associated to Δ , and denote by c_{ij}^k the structural constant of the canonical frame $\{X_i\}_{i=1}^3$. By definition, $c_{ij}^k = -c_{ji}^k$. Moreover, (10) and (11) imply

$$(17) \quad c_{12}^3 = 1, \quad c_{13}^3 = c_{23}^3 = 0.$$

More symmetry relations of the structural constants can be obtained from the Jacobi identity

$$[X_1, [X_2, X_3]] + [X_3, [X_1, X_2]] + [X_2, [X_3, X_1]] = 0.$$

In particular, if M is a Lie group, the structural constants do not depend on the points of the base manifold M , and the Jacobi identity is equivalent to the following relations:

$$(18) \quad c_{13}^1 + c_{23}^2 = 0, \quad c_{12}^1 c_{13}^1 + c_{12}^2 c_{23}^1 = 0, \quad c_{12}^1 c_{13}^2 + c_{12}^2 c_{23}^2 = 0.$$

Let $\nu \in \Delta$ be a horizontal normal of a regular hyper-surface $W \subset M$. Taking into account (17), we write the Δ -minimal surface equation at non-characteristic points:

$$(19) \quad \left(\text{div}^\Delta \nu + \nu_1 c_{12}^2 - \nu_2 c_{12}^1 \right) \Big|_W = 0.$$

If W is a level set of some smooth function F , using (6), we obtain

$$(20) \quad \left[\left(X_1^2 F (X_2 F)^2 + X_2^2 F (X_1 F)^2 - X_1 F X_2 F (X_1 \circ X_2 + X_2 \circ X_1) F \right) D_1^{-3} + \right. \\ \left. + \left(c_{12}^2 X_1 F - c_{12}^1 X_2 F \right) D_1^{-1} \right] \Big|_W = 0, \quad D_1 = \sqrt{X_1 F^2 + X_2 F^2}.$$

The last equation is a highly degenerate PDE. Many non-trivial solutions of this equation are known for Δ^{H^1} , the interested reader can consult in [4] and other papers, cited in the Bibliography. Another important for applications case is the distribution, which corresponds to another Lie Group, the so-called *roto-translational group* e^2 .

Example 2 The Lie group e^2 can be realized as $\mathbb{R}^2 \times \mathbb{S}^1$ with local coordinates (x, y, z) . The Lie algebra corresponding to this group is generated by vector fields

$$X_1 = \cos z \partial_x + \sin z \partial_y, \quad X_2 = \partial_z.$$

It is easy to check that the horizontal distribution Δ^{e^2} with sections $\Delta_q^{e^2} = \text{span}\{X_1(q), X_2(q)\}$, $q \in M$, is contact, the corresponding canonical 1-form is $\omega = \pm(\sin z dx - \cos z dy)$. The Reeb vector field coincides with the Lie bracket $[X_1, X_2]$ (up to the sign) and the only non-zero structural constants are $c_{12}^3 = c_{23}^1 = \pm 1$. The following surfaces are Δ^{e^2} -minimal surfaces (away from the characteristic points):

- a). $y = x + B(\sin z + \cos z) + C$, $B, C = \text{const}$;
- b). $Ax + B \sin z = C$, $A, B, C = \text{const}$;
- c). $x \cos z + y \sin z = 0$.

3.1 Structure of characteristic points

In [4] the authors showed (see Theorem B) that in the case of the Heisenberg distribution Δ^{H^1} the characteristic points of the corresponding minimal surfaces are either isolated of index +1 or contained in a C^1 curve. Such curves are called singular. The characteristic curves keep go straight after they cross a singular curve. All these facts are of local nature, and it turns out that they hold true for Sub-Riemannian minimal surfaces in generic contact Sub-Riemannian manifolds of dimension 3. Basically all local arguments used in [4] can be directly applied in the general case modulo a suitable choice of local coordinates in M in the small neighborhood of a characteristic point.

Let $F \in C^2(M)$ and let W be a Δ -minimal smooth surface in M defined as a level set of F , i.e., assume F satisfies (20) away from the points $\hat{q} \in W$ where $X_1 F(\hat{q}) = X_2 F(\hat{q}) = 0$.

Let \hat{q} be a characteristic point of W . Since $d_{\hat{q}} F \neq 0$ one can choose the local coordinates in M in a small neighborhood $\mathcal{O}_{\hat{q}} = \{q = (x, y, z) \in \mathbb{R}^3\}$ in such a way that $W = \{q; F(q) \equiv z = 0, q \in \mathcal{O}_{\hat{q}}\}$. Since \hat{q} is a characteristic point $\Delta_{\hat{q}} = T_{\hat{q}} W = \{(x, y, 0)\}$. Then

$$X_1(x, y, z) = \partial_x + a_1(x, y, z) \partial_z,$$

$$(21) \quad X_2(x, y, z) = xa_3(x, y, z)\partial_x + \partial_y + a_2(x, y, z)\partial_z,$$

$$a_i(0, 0, 0) = 0 \quad \text{for } i = 1, 2,$$

where a_1 , a_2 and a_3 are some functions, which define the Sub-Riemannian structure on Δ . In particular, since Δ is bracket generating at any point

$$\text{rank}\{X_1(\hat{q}), X_2(\hat{q}), [X_1, X_2](\hat{q})\} = 3,$$

and so

$$(22) \quad \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_3(0) & 0 & \frac{\partial a_2}{\partial x}(0) - \frac{\partial a_1}{\partial y}(0) \end{pmatrix} = \frac{\partial a_2}{\partial x}(0) - \frac{\partial a_1}{\partial y}(0) \neq 0.$$

Denote

$$A = \begin{pmatrix} \frac{\partial a_1}{\partial x} & \frac{\partial a_1}{\partial y} \\ \frac{\partial a_2}{\partial x} & \frac{\partial a_2}{\partial y} \end{pmatrix} (0) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The matrix A plays a key role on the further analysis. First of all, condition (22) means that b and c cannot be zero simultaneously. Thus A is not a 0-matrix and $\text{trace} A \neq 0$. Moreover, the implicit function theorem implies that $\det A = 0$ if and only if the point \hat{q} is not isolated and contained in a C^1 curve called singular. In particular, if \hat{q} is isolated, then $\det A \neq 0$. If \hat{q} is not isolated, the characteristic curves keep go straight through a singular curve. The proof of these facts is rather technical and it repeats the proof of the analog result in the Heisenberg group (see [4]) with a few changes due to the use of the new curvilinear coordinates (x, y, z) . We omit this proof here, an interested reader can find it in the original paper. Let us just show explicitly that the index of an isolated characteristic point of a Δ -minimal surface is equal to $+1$ and it not affected by the difference of the given Sub-Riemannian structure on Δ from the Heisenberg one.

By definition, the characteristic point \hat{q} is the singular point of the vector field $\nu_0 = X_1 F X_1 + X_2 F X_2$ and $\text{ind}(\hat{q}) = \text{sgn } \det A$. Denote $H = X_1^2 F (X_2 F)^2 + X_2^2 F (X_1 F)^2 - X_1 F X_2 F (X_1 \circ X_2 + X_2 \circ X_1) F$. Then equation (20) becomes

$$(23) \quad \frac{H + (X_1 F^2 + X_2 F^2)(c_{12}^2 X_1 F - c_{12}^1 X_2 F)}{D_1^3} = 0,$$

Let $q = \hat{q} + \delta q$, where $\delta q = (\delta x, \delta y, 0)$ and $r = \sqrt{\delta x^2 + \delta y^2}$. By (21), we have

$$\begin{aligned} X_1 F(q) &= a\delta x + b\delta y + o(r), & X_2 F(\hat{q}) &= c\delta x + d\delta y + o(r), \\ X_1^2 F(q) &= a, & X_2^2 F(q) &= d, & (X_1 \circ X_2 + X_2 \circ X_1)F(q) &= c + b, \\ D_1^2(q) &= (X_1 F^2 + X_2 F^2)(q) = \\ &= (a^2 + c^2)\delta x^2 + (b^2 + d^2)\delta y^2 + 2(ab + cd)\delta x\delta y + o(r^2). \end{aligned}$$

Substituting these expressions into (23) we obtain

$$\begin{aligned} &(H + (X_1 F^2 + X_2 F^2)(c_{12}^2 X_1 F - c_{12}^1 X_2 F))(q) = \\ &= a(c\delta x + d\delta y + o(r))^2 + d(a\delta x + b\delta y + o(r))^2 - \\ &\quad - (c + b)(a\delta x + b\delta y + o(r))(c\delta x + d\delta y + o(r)) + \\ &\quad + (c_{12}^2(\hat{q} + \delta q)(a\delta x + b\delta y + o(r)) - c_{12}^1(\hat{q} + \delta q)(c\delta x + d\delta y + o(r))) \times \\ &\quad \times ((a^2 + c^2)\delta x^2 + (b^2 + d^2)\delta y^2 + 2(ab + cd)\delta x\delta y + o(r^3)) = \\ &= (ad - bc)(a\delta x^2 + d\delta y^2 + (c + b)\delta x\delta y) + o(r^2). \end{aligned}$$

We see that the structural constants c_{12}^1 and c_{12}^2 enters into the play together with the higher order terms and do not affect the type of characteristic points. Further, since \hat{q} is isolated $\det A = ad - bc \neq 0$. Moreover, by (23), we have

$$\frac{a\delta x^2 + d\delta y^2 + (c + b)\delta x\delta y + o(r^2)}{((a^2 + c^2)\delta x^2 + (b^2 + d^2)\delta y^2 + 2(ab + cd)\delta x\delta y + o(r^2))^{3/2}} = 0.$$

Observe that if we take $\delta x = 0$, in order to satisfy the last equation we should necessarily have $d = 0$, analogously $\delta y = 0$ forces $a = 0$. This implies $b + c = 0$ and hence $b = -c$. Therefore $\det A = -bc = c^2$ and hence $\text{ind}(\hat{q}) = +1$.

So, locally the Δ^{H^1} -minimal surfaces give a good approximation of the structure of generic Δ -minimal surfaces. But globally this is not true. We will show this difference in the next subsection.

3.2 Characteristic curves

Recall that the curves formed by the intersection of the distribution Δ with W are called *the characteristic curves* of the Δ -minimal surface W . These curves are the integral curves of *the characteristic vector field* $e \in \Delta$ such that $\langle e, \nu \rangle_\Delta = 0$ and $\|e\|_\Delta = 1$. Thus $e = e_1 X_1 + e_2 X_2$ and we fix the orientation on Δ by setting $e_1 = \nu_2$, $e_2 = -\nu_1$. Since $\|e\|_\Delta = 1$ one can introduce an auxiliary parameter $\phi \in S^1$ such that $\cos \phi = e_1$, $\sin \phi = e_2$, and

$$e \equiv e^\phi = \cos \phi X_1 + \sin \phi X_2.$$

Here we use the upper-index ϕ to stress out the dependence of the vector field e on ϕ . Now equation (19) can be rewritten as follows:

$$(24) \quad -\sin \phi X_2 \phi - \cos \phi X_1 \phi = \cos \phi c_{12}^1 + \sin \phi c_{12}^2.$$

Equation (24) is a quasilinear PDE and one can apply the classical method of characteristics to find its solutions. Indeed, let $s \mapsto (q_1(s), q_2(s), q_3(s))$ be a smooth (at least C^1) curve in M . Along this curve $\dot{\phi} = \sum_{i=1}^3 \frac{\partial \phi}{\partial q_i} \dot{q}_i$ with $\dot{\cdot} = \frac{d}{ds}$. Then (19) is equivalent to the following system of the first order ODE:

$$(25) \quad \begin{cases} \dot{q} &= e^\phi(q) \\ \dot{\phi} &= -\cos \phi c_{12}^1(q) - \sin \phi c_{12}^2(q) \end{cases}$$

This shows that the Δ -minimal surface W is nothing but the projection on M of the integral surface of the system (25) in the extended manifold $M_0 = M \times S^1 = \{(q, \phi) : q \in M, \phi \in S^1\}$. Thus, at least locally, one can find solutions of the Δ -minimal surface equation by solving the Cauchy problem for the system (25). The characteristic points of W , being the points where (9) is not defined, are either the singular points of the surface (25) in the extended space, or the singular points of the projection $\pi : M_0 \mapsto M$, for instance, the points where the different characteristics meet each other.

In a particular but important for the applications case of Lie groups system (25) can be integrated explicitly, at least formally. Indeed, if c_{12}^1 and c_{12}^2 are constant, the second equation of (25) can be integrated separately and then the obtained function $\phi(t)$ can be used to integrate the first equation of (25). Moreover, if

$$(26) \quad c_{12}^1(q) = c_{12}^2(q) = 0 \quad \forall q \in M,$$

then ϕ is constant along any characteristic curve. The corresponding minimal surface is a kind of ruled surface, whose rulings are the characteristic curves that are not straight lines in general.

Example 3 For the Heisenberg-type distribution Δ^{H^1} condition (26) is always satisfied. Thus ϕ is constant along characteristic curves. The characteristic vector field reads

$$e^\phi = \cos \phi \partial_x + \sin \phi \partial_y + \frac{1}{2}(x \sin \phi - y \cos \phi) \partial_z, \quad \phi \in [0, 2\pi].$$

Thus any characteristic curve satisfies the following system of ODE for some fixed $\phi \in [0, 2\pi]$:

$$(27) \quad \begin{cases} \dot{x} &= \cos \phi \\ \dot{y} &= \sin \phi \\ \dot{z} &= \frac{1}{2}(x \sin \phi - y \cos \phi) \end{cases}$$

The solution of this system is the curve

$$\begin{cases} x &= t \cos \phi + x_0 \\ y &= t \sin \phi + y_0 \\ z &= \frac{1}{2}(x_0 \sin \phi - y_0 \cos \phi)t + z_0 \end{cases},$$

starting at $(x, y, z)(0) = (x_0, y_0, z_0)$. We immediately see that the characteristic curves of Δ^{H^1} -minimal surfaces lie on straight lines. This fact was first noticed in [4] and it has a lot of important consequences for the global structure of minimal surfaces in H^1 . For example, all Δ^{H^1} -minimal surfaces are standard ruled surfaces. The fact that the characteristic curves are straight lines implies that any Δ^{H^1} -minimal surface can contain at most one isolated characteristic point. In Figure 1 there are shown some examples of Δ^{H^1} -minimal surfaces, obtained by integration of system (25) forward and backward in time with help of Mathematica. The fat line denotes the curve of initial conditions $\gamma(s) = q_s(0)$ parametrized by some auxiliary parameter s .

Example 4 In the case of group of roto-translations (see Example 2) condition (26) is satisfied as well, and ϕ is constant along characteristics. For any fixed $\phi \in [0, 2\pi]$ the characteristic vector field is given by $e^\phi = \cos \phi \cos z \partial_x + \cos \phi \sin z \partial_y + \sin \phi \partial_z$. Let us find explicitly the characteristic curves. They satisfy the following system of ODE:

$$(28) \quad \begin{cases} \dot{x} &= \cos \phi \cos z \\ \dot{y} &= \cos \phi \sin z \\ \dot{z} &= \sin \phi \end{cases}$$

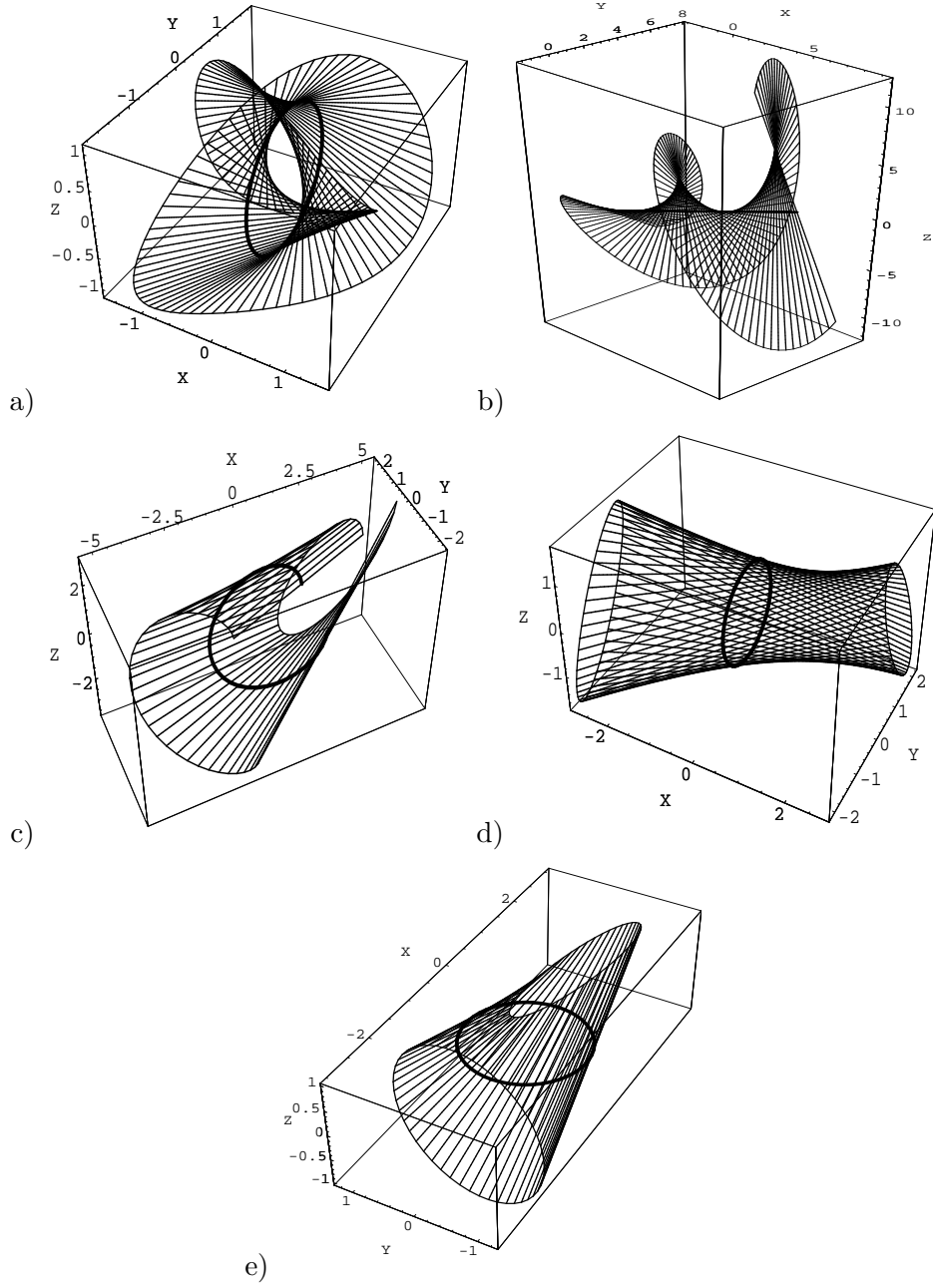


Figure 1: Examples of Δ^{H^1} -minimal surfaces: a). $\gamma(s) = (0, \cos s, \sin s)$, $\phi(s) = s$, $s \in [0, 2\pi]$, $t \in [-\frac{3}{2}, \frac{3}{2}]$; b) $\gamma(s) = (s, s, 0)$, $\phi(s) = s$, $s \in [0, 2\pi]$, $t \in [-3, 3]$; c). $\gamma(s) = (2 \cos s, 1 - \frac{s}{\pi}, 2 \sin s)$, $\phi = \frac{\pi}{8}$, $t \in [-1, 1]$; d). $\gamma(s) = (0, \cos s, \sin s)$, $\phi = \frac{\pi}{8}$, $t \in [-2, 2]$; e). $\gamma(s) = (\cos s, \sin s, 0)$, $\phi = \frac{\pi}{25}$, $t \in [-2, 2]$.

The solution that starts at a point (x_0, y_0, z_0) has the form

$$\begin{cases} x &= \frac{\cos \phi}{\sin \phi} \sin z + x_0 \\ y &= -\frac{\cos \phi}{\cos \phi} \sin z + y_0 \\ z &= t \sin \phi + z_0 \end{cases} \quad \text{for } \phi \neq 0, \pi,$$

or

$$\begin{cases} x &= \pm t \cos z_0 + x_0 \\ y &= \pm t \sin z_0 + y_0 \\ z &= z_0 \end{cases} \quad \text{for } \phi = 0, \pi.$$

In the latter case the characteristic curves are straight lines, though in general this is not true. In Figure 2 we present some examples of Δ^{e^2} -minimal surfaces, constructed by solving numerically equations (25) forward and backward in time by Mathematica. As before, the fat line denotes the curve of initial conditions $\gamma(s) = q_s(0)$.

3.3 Characteristic curves of Sub-Riemannian minimal surfaces and Sub-Riemannian geodesics

It is natural to ask whether the characteristic curves of Sub-Riemannian minimal surfaces are Sub-Riemannian geodesics. Recall that Sub-Riemannian geodesics are horizontal curves $t \mapsto \eta(t)$, $\dot{\eta}(t) \in \Delta_{\eta(t)}$, $t \in [0, T]$, which minimize the Sub-Riemannian length

$$\ell[\eta] = \int_0^T \|\dot{\eta}(\tau)\|_{\Delta} d\tau$$

and such that $\|\dot{\eta}(t)\|_{\Delta}$ is constant for all $t \in [0, T]$ (see [3]). The existence of Sub-Riemannian geodesics is guaranteed by the Hopf-Rinow theorem provided the distribution Δ is bracket-generating. According to the classical Pontryagin Maximum Principle the Sub-Riemannian geodesics are projections on the base manifold M of the corresponding Pontryagin extremals in T^*M . In the contact case all these extremals are integral curves of the Hamiltonian vector field $\vec{h} \in \text{Vec}(T^*M)$ ³:

$$(29) \quad \vec{h} = u_1 X_1 + u_2 X_2 + a \partial_{u_3} - (u_3 + b) \partial_{12},$$

where

$$u_i(p, q) = \langle p, X_i(q) \rangle, \quad q \in M, p \in T_q^* M, i = 1, 2, 3,$$

³For the details see [1], [2].

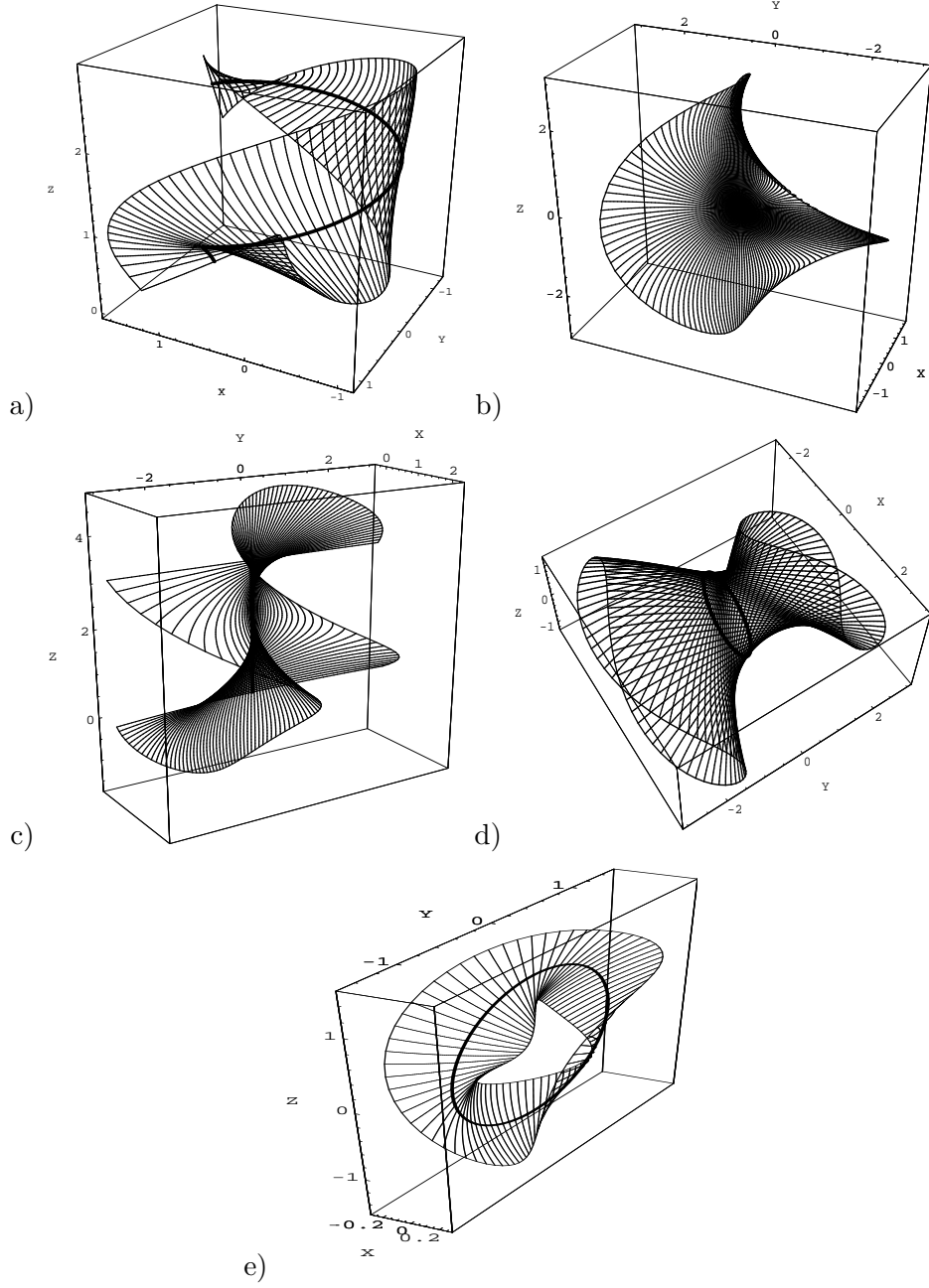


Figure 2: Examples of Δ^{e^2} -minimal surfaces: a) $\gamma(s) = (\cos s, \sin s, \sqrt{s})$, $\phi(s) = \frac{s}{2}$, $s \in [0, 2\pi]$, $t \in [-1, 1]$; b) $\gamma(s) = (0, 0, 0)$, $\phi(s) = s$, $s \in [0, 2\pi]$, $t \in [0, 3]$; c) $\gamma(s) = (0, 0, s)$, $\phi(s) = s$, $s \in [0, \pi]$, $t \in [-3, 3]$; d) $\gamma(s) = (\cos s, 0, \sin s)$, $\phi(s) = \frac{2\pi}{75}s$, $s \in [0, 2\pi]$, $t \in [-3, 3]$; e) $\gamma(s) = (0, \cos s, \sin s)$, $\phi(s) = s$, $s \in [0, 2\pi]$, $t \in [-\frac{1}{2}, \frac{1}{2}]$.

$$\begin{aligned}
\partial_{12} &= u_1 \partial_{u_2} - u_2 \partial_{u_1}, \\
a = \vec{h}(u_3) &= c_{31}^1(u_1^2 - u_2^2) + (c_{32}^1 + c_{31}^2)u_1 u_2 \\
b &= u_1 c_{12}^1 + u_2 c_{12}^2,
\end{aligned}$$

and

$$u_1^2 + u_2^2 = 1.$$

The last condition permits us to introduce a coordinate $\psi \in S^1$ on the oriented circle such that $u_1 = \cos \psi$, $u_2 = \sin \psi$. Then $\partial_{12} \equiv \partial_\psi$. In particular, it follows that

$$(30) \quad \dot{\psi} = -u_3 - b, \quad \dot{u}_3 = a.$$

The characteristic curve of a Δ -minimal surface W is a Sub-Riemannian geodesic if and only if

$$(31) \quad \pi_*(\vec{h}) = e^\phi,$$

i.e., $u_1 X_1 + u_2 X_2 = e^\phi$. Thus we can identify ψ and ϕ . Now comparing (30) with (25) we see that the characteristic curves are Sub-Riemannian geodesics provided $u_3 = 0$, which implies $a = 0$, i.e.,

$$(32) \quad c_{31}^1 \cos 2\phi + \frac{c_{32}^1 + c_{31}^2}{2} \sin 2\phi = 0.$$

The direct computation yields the solutions of this equation:

$$(33) \quad \phi^*(q) = \begin{cases} -\frac{1}{2} \arctan \left(\frac{2c_{13}^1(q)}{c_{23}^1(q) + c_{13}^2(q)} \right) + \frac{k\pi}{2} & \text{if } c_{32}^1(q) + c_{31}^2(q) \neq 0; \\ \frac{(2k+1)\pi}{4} & \text{if } c_{32}^1(q) + c_{31}^2(q) = 0, \\ & c_{13}^1(q) \neq 0, \end{cases}$$

where $k = 0, 1, 2, 3$. The characteristic curves that are Sub-Riemannian geodesics satisfy the ODE $\dot{q} = e^{\phi^*(q)}(q)$. If both coefficients in (32) are zero, then all characteristic curves are Sub-Riemannian geodesics, parametrized according to (30).

In the particular case of Lie groups, the structural constants c_{ij}^k do not depend on the point of the base manifold. Hence $\phi^* = \text{const}$ and we have an additional condition:

$$(34) \quad b = c_{12}^1 \cos \phi + c_{12}^2 \sin \phi = 0.$$

In general, this condition is stronger than (32). It is easy to check that if (34) is non-degenerate, its solutions belong to the set of solutions of (32).

Indeed, if $c_{12}^2 \neq 0$, then combining (32) and (34) by direct computation we obtain the following compatibility condition for the structural constants:

$$(35) \quad c_{13}^1((c_{12}^1)^2 - (c_{12}^2)^2) + c_{12}^1 c_{12}^2 (c_{23}^1 + c_{13}^2) = 0.$$

Comparing (35) with (18) one can see that it is trivially satisfied. If $c_{12}^1 \neq 0$, $c_{12}^2 = 0$, then (18) implies $c_{13}^1 = c_{13}^2 = 0$ and (32) reduces to $c_{23}^1 \sin 2\phi = 0$. On the other hand, in this case (34) yields $\cos \phi = 0$, which clearly satisfy (32). Summing up, we obtain the following classification:

- a). if $c_{12}^2 \neq 0$, then $\phi^* = -\arctan\left(\frac{c_{12}^1}{c_{12}^2}\right) + k\pi$, $k = 0, 1$;
- b). if $c_{12}^2 = 0$, then $\phi^* = \frac{k\pi}{2}$, $k = 1, 3$;
- c). if $c_{12}^1 = c_{12}^2 = 0$ then ϕ^* is given by (33);
- d). all characteristic curves are Sub-Riemannian geodesics if both (32) and (34) degenerate.

We conclude this discussion by analysis of the characteristic curves in H^1 and e^2 .

Example 5 In the Heisenberg case, since all structural constants, but c_{12}^3 , vanish, the parameter ϕ^* can take any value in $[0, 2\pi]$. As we have already seen, the characteristic curves are straight lines and they all are Sub-Riemannian geodesics. Fixing a point $q_0 \in M$ and varying $\phi^* \in [0, 2\pi]$ one can generate a plane, which is a totally geodesic surface in the Sub-Riemannian sense, and in the same time it is an entire Δ^{H^1} -minimal surface with one characteristic point at q_0 .

Example 6 In the case of the distribution Δ^{e^2} there are two non-zero structural constants $c_{12}^3 = c_{23}^1 = 1$. From (32) we obtain $\phi^* = \frac{k\pi}{2}$, $k = 1, 3$. A simple calculation shows that there are two families of characteristic curves that are Sub-Riemannian geodesics:

$$x = t \cos z_0 + x_0, \quad y = t \sin z_0 + y_0 \quad z = z_0;$$

$$x = x_0, \quad y = y_0, \quad z = \pm t + z_0.$$

Note that these curves are actually straight lines.

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